

Luca Salasnich and Flavio Toigo

# Viscosity-entropy ratio of the unitary Fermi gas from zero-temperature elementary excitations

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**Abstract** We investigate the low-temperature behavior of the ratio between the shear viscosity  $\eta$  and the entropy density  $s$  in the unitary Fermi gas by using a model based on the zero-temperature spectra of both bosonic collective modes and fermionic single-particle excitations. Our theoretical curve of  $\eta/s$  as a function of the temperature  $T$  is in qualitative agreement with the experimental data of trapped ultracold  $^6\text{Li}$  atomic gases. We find the minimum value  $\eta/s \simeq 0.44$  (in units of  $\hbar/k_B$ ) at the temperature  $T/T_F \simeq 0.27$ , with  $T_F$  the Fermi temperature.

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## 1 Introduction

Strongly interacting quantum many-body systems like Helium 4, the quark-gluon plasma and the unitary Fermi gas share a common feature: an extremely low viscosity hydrodynamics.<sup>1,2</sup> These quite different many-body systems show a ratio of shear viscosity  $\eta$  to entropy density  $s$  which is not too far from the lower bound  $\eta/s = \hbar/(4\pi k_B)$  predicted for a “perfect fluid” by using the anti-deSitter/conformal field theory (AdS/CFT) duality between certain strongly coupled field theories in  $d = 4$  space-time dimensions and weakly coupled string theory in  $d = 10$ .<sup>3</sup> As discussed in a recent review,<sup>7</sup> theoretical predictions<sup>5,6</sup> of the viscosity-entropy ratio for dilute and ultracold Fermi atoms in the unitary regime, where the s-wave inter-atomic scattering length  $a_F$  diverges, are not in good agreement with the experimental data of the viscosity-entropy ratio measured in the  $^6\text{Li}$  atomic gas.<sup>4</sup>

In this paper we study the low-temperature behavior of  $\eta/s$  by using a recent heuristic analysis of the shear viscosity<sup>8</sup> and a thermodynamical model<sup>9</sup> of the

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Dipartimento di Fisica “Galileo Galilei” and CNISM, Università di Padova, Via Marzolo 8, 35122 Padova, Italy  
E-mail: luca.salasnich@unipd.it, flavio.toigo@unipd.it

unitary Fermi gas based on zero-temperature elementary excitations. We show that our theoretical curve for  $\eta/s$  as a function of the temperature  $T$  is in qualitative good agreement with the experimental data of trapped ultracold  $^6\text{Li}$  atomic gases. In particular, we find the minimum value  $\eta/s \simeq 0.44$  (in units of  $\hbar/k_B$ ) at the temperature  $T/T_F \simeq 0.27$ , with  $T_F$  the Fermi temperature. Both the value and the position of this minimum are fully compatible with the most recent experimental determinations.<sup>2,4</sup>

In the first part of this paper we briefly review our thermodynamical model<sup>9</sup> of the unitary Fermi gas comparing it with experimental data<sup>10</sup> and Monte Carlo simulations<sup>11</sup>. In the second part we adopt the analysis of How and LeClair<sup>8</sup> for the shear viscosity and derive from it and from our thermodynamical model<sup>9</sup> the viscosity-entropy ratio  $\eta/s$ . We then compare our curve of  $\eta/s$  vs  $T$  with available experimental data<sup>4</sup> and proposed theories.<sup>5,6,7,8</sup>

## 2 Elementary excitations of the unitary Fermi gas

For any many-body system the weakly excited states, the so-called elementary excitations, can be treated as excitations of an ideal gas.<sup>12,13</sup> In general, these elementary excitations are the result of collective interactions of the particles of the system, and therefore pertain to the system as a whole and not to its separate particles.<sup>12,13</sup> For the unitary Fermi gas the mean-field extended BCS theory predicts the existence of fermionic single-particle elementary excitations characterized by an energy gap  $\Delta$ .<sup>14</sup> The inclusion of beyond-mean-field effects, namely quantum fluctuations of the order parameter, gives rise to bosonic collective excitations,<sup>14</sup> which are density waves reducing to the Bogoliubov-Goldstone-Anderson mode in the limit of small momenta.<sup>15</sup>

Our effective quantum Hamiltonian<sup>9</sup> of the uniform unitary Fermi gas with two equally-populated spin components is then assumed to be:

$$\hat{H} = E_0 + \sum_{\mathbf{q}} \varepsilon_{col}(\mathbf{q}) \hat{b}_{\mathbf{q}}^{\dagger} \hat{b}_{\mathbf{q}} + \sum_{\sigma=\uparrow,\downarrow} \sum_{\mathbf{p}} \varepsilon_{sp}(\mathbf{p}) \hat{c}_{\mathbf{p}\sigma}^{\dagger} \hat{c}_{\mathbf{p}\sigma}, \quad (1)$$

where  $E_0$  is the ground-state energy,  $\hat{b}_{\mathbf{q}}^{\dagger}$  and  $\hat{b}_{\mathbf{q}}$  are the bosonic creation and destruction operators of a collective excitation of linear momentum  $\mathbf{q}$  with energy  $\varepsilon_{col}(\mathbf{q})$ , while  $\hat{c}_{\mathbf{p}\sigma}^{\dagger}$  and  $\hat{c}_{\mathbf{p}\sigma}$  are the fermionic creation and destruction operators of a single-particle excitation of linear momentum  $\mathbf{p}$  and spin  $\sigma$ , with energy  $\varepsilon_{sp}(\mathbf{p})$ .

It is now well-established<sup>14</sup> that the ground-state energy  $E_0$  of the uniform unitary Fermi gas made of  $N$  atoms in a volume  $V$  is given by

$$E_0 = \frac{3}{5} \xi N \varepsilon_F, \quad (2)$$

with  $\xi \simeq 0.4$ <sup>16</sup> and where  $\varepsilon_F = \hbar^2 (3\pi^2 n)^{2/3} / (2m)$  is the Fermi energy of a non-interacting fermi gas with density  $n = N/V$ .

The exact dispersion relation of elementary (collective and single-particle) excitations is not fully known.<sup>14</sup> In Ref.<sup>15</sup> we have found the dispersion relation of collective elementary excitations as

$$\varepsilon_{col}(\mathbf{q}) = \sqrt{c_1^2 q^2 + \frac{\lambda}{4m^2} q^4}, \quad (3)$$

where

$$c_1 = \sqrt{\frac{\xi}{3}} v_F, \quad (4)$$

is the zero-temperature first sound velocity, with  $v_F = (\hbar/m)(3\pi^2 n)^{1/3}$  the Fermi velocity of a noninteracting Fermi gas. Notice that the term with  $\lambda$  takes into account the increase of kinetic energy due to spatial variations of the density<sup>15,17,18,19,20,21,22</sup>. For the purposes of the present paper, by fixing  $\xi = 0.42$ , i.e. the value given by the Monte Carlo prediction for a uniform gas of Astrakharchik *et al.*,<sup>23</sup> we find that the best agreement with Monte Carlo data is obtained with  $\lambda = 0.25$ .

The collective modes describe correctly only the low-energy density oscillations of the system while at higher energies one expects the appearance of fermionic single-particle excitations starting from the threshold above which there is the breaking of Cooper pairs.<sup>14,11,24</sup> At zero temperature these single-particle elementary excitations can be written as

$$\varepsilon_{sp}(p) = \sqrt{\left(\frac{p^2}{2m} - \xi \varepsilon_F\right)^2 + \Delta_0^2} \quad (5)$$

where  $\xi$  is a parameter which takes into account the interaction between fermions ( $\xi \simeq 0.9$  according to recent Monte Carlo results<sup>24</sup>) with  $\varepsilon_F$  the Fermi energy of the ideal Fermi gas.  $\Delta_0$  is the zero-temperature gap parameter with  $2\Delta_0$  the minimal energy to break a Cooper pair.<sup>14</sup> Notice that the gap energy  $\Delta_0$  of the unitary Fermi gas at zero-temperature has been calculated with Monte Carlo simulations<sup>24,25</sup> and found to be  $\Delta_0 = \gamma \varepsilon_F$ , with  $\gamma \simeq 0.45$ .

### 3 Thermodynamics of the unitary Fermi gas

At very low temperature the thermodynamic properties of the superfluid unitary Fermi gas can be obtained from the collective spectrum and considering it as an ideal Bose gas of elementary excitations<sup>12</sup> with the bosonic distribution

$$f_B(q) = \langle \hat{b}_{\mathbf{q}}^+ \hat{b}_{\mathbf{q}} \rangle = \frac{1}{e^{\varepsilon_{col}(q)/k_B T} - 1}, \quad (6)$$

where  $\langle \hat{A} \rangle = \text{Tr}[\hat{A} e^{-\hat{H}/k_B T}] / \text{Tr}[e^{-\hat{H}/k_B T}]$  is the thermal average of the operator  $\hat{A}$  with  $T$  the absolute temperature and  $k_B$  is the Boltzmann constant.<sup>26</sup> As  $T$  increases also the fermionic single-particle excitations become important. Thus there is also the effect of an ideal Fermi gas of single-particle excitations with the fermionic distribution

$$f_F(p) = \langle \hat{c}_{\mathbf{p}\sigma}^+ \hat{c}_{\mathbf{p}\sigma} \rangle = \frac{1}{e^{\varepsilon_{sp}(p)/k_B T} + 1}, \quad (7)$$

which is spin independent.

The Helmholtz free energy  $F$  of any thermodynamic system is given by

$$F = -k_B T \ln \mathcal{Z}, \quad (8)$$

where

$$\mathcal{Z} = \text{Tr}[e^{-\hat{H}/k_B T}] , \quad (9)$$

is the partition function of the system.<sup>26</sup> Using Eq. (1) the free energy of our unitary Fermi gas can be written as  $F = F_0 + F_{col} + F_{sp}$ , where  $F_0$  is the free energy of the ground-state,  $F_{col}$  is the free energy of the bosonic collective excitations and  $F_{sp}$  the free energy of fermionic single-particle excitations. The Helmholtz free energy  $F_0$  of the uniform ground state coincides with the zero-temperature internal energy  $E_0$  and is given by

$$F_0 = \frac{3}{5} \xi N \varepsilon_F , \quad (10)$$

where  $N$  is the number of atoms of the uniform system in a volume  $V$ . The free energy  $F_{col}$  of the collective excitations is instead given by<sup>12</sup>

$$F_{col} = k_B T \sum_{\mathbf{q}} \ln \left[ 1 - e^{-\varepsilon_{col}(\mathbf{q})/(k_B T)} \right] , \quad (11)$$

while the free energy  $F_{sp}$  due to the single-particle excitations is

$$F_{sp} = -2 k_B T \sum_{\mathbf{p}} \ln \left[ 1 + e^{-\varepsilon_{sp}(\mathbf{p})/(k_B T)} \right] , \quad (12)$$

where the factor 2 is due to the two spin components. As previously discussed, the total Helmholtz free energy  $F$  of the low-temperature unitary Fermi gas can be then written as  $F_0 + F_{col} + F_{sp}$ , namely

$$F = N \varepsilon_F \Phi \left( \frac{T}{T_F} \right) , \quad (13)$$

where  $\Phi(x)$  is a function of the scaled temperature  $x = T/T_F$ , with  $T_F = \varepsilon_F/k_B$ , given by

$$\begin{aligned} \Phi(x) = & \frac{3}{5} \xi + \frac{3}{2} x \int_0^{+\infty} \ln \left[ 1 - e^{-\tilde{\varepsilon}_{col}(u)/x} \right] u^2 du \\ & - 3x \int_0^{+\infty} \ln \left[ 1 + e^{-\tilde{\varepsilon}_{sp}(u)/x} \right] u^2 du . \end{aligned} \quad (14)$$

Here the discrete summations have been replaced by integrals, moreover we set  $\tilde{\varepsilon}_{col}(u) = \sqrt{u^2(\lambda u^2 + 4\xi/3)}$  and  $\tilde{\varepsilon}_{sp}(u) = \sqrt{(u^2 - \xi)^2 + \gamma^2}$ .

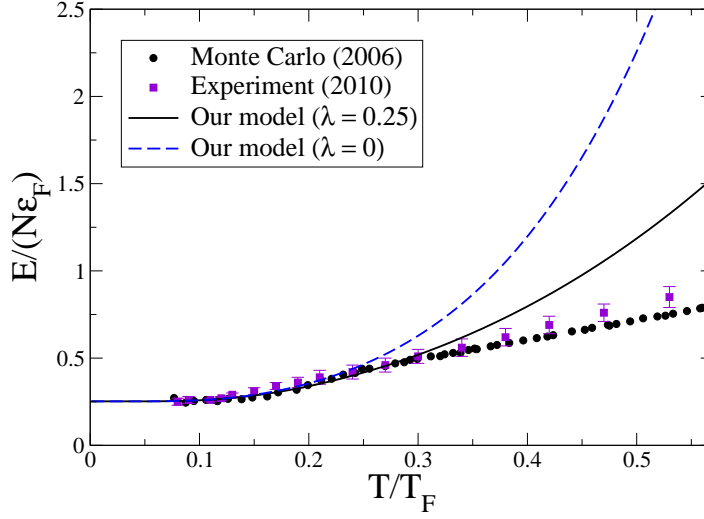
From the Helmholtz free energy  $F$  we can immediately calculate the chemical potential  $\mu$ , through

$$\mu = \left( \frac{\partial F}{\partial N} \right)_{T,V} . \quad (15)$$

obtaining

$$\mu = \varepsilon_F \left[ \frac{5}{3} \Phi \left( \frac{T}{T_F} \right) - \frac{2}{3} \frac{T}{T_F} \Phi' \left( \frac{T}{T_F} \right) \right] , \quad (16)$$

where  $\Phi'(x) = \frac{d\Phi(x)}{dx}$  and one recovers  $\mu_0 = \xi \varepsilon_F$  in the limit of zero-temperature.



**Fig. 1** (Color online). Scaled internal energy  $E/(N\varepsilon_F)$  as a function of the scaled temperature  $T/T_F$ . Filled circles: Monte Carlo simulations<sup>11</sup>. Open squares with error bars: experimental data of Horikoshi *et al.*<sup>10</sup>. Solid line: our model, i.e. Eq. (20) and Eq. (14), with  $\lambda = 0.25$ . Solid line: our model, i.e. Eq. (20) and Eq. (14), with  $\lambda = 0$ . Other zero-temperature parameters of elementary excitations:  $\xi = 0.42$ ,  $\zeta = 0.9$ , and  $\gamma = 0.45$ .

The entropy  $S$  is related to the free energy  $F$  by the formula

$$S = - \left( \frac{\partial F}{\partial T} \right)_{N,V}, \quad (17)$$

from which we get

$$S = -Nk_B \Phi' \left( \frac{T}{T_F} \right). \quad (18)$$

In addition, the internal energy  $E$ , given by

$$E = F + TS, \quad (19)$$

can be written explicitly as

$$E = N\varepsilon_F \left[ \Phi \left( \frac{T}{T_F} \right) - \frac{T}{T_F} \Phi' \left( \frac{T}{T_F} \right) \right]. \quad (20)$$

It is interesting to compare our model with other theoretical approaches and also with the available experimental data. In Fig. 1 we report the data of internal energy  $E$  obtained by Bulgac, Drut and Magierski<sup>11</sup> with their Monte Carlo simulations (filled circles) of the atomic unitary gas. We insert also the very recent experimental data of Horikoshi *et al.*<sup>10</sup> for the unitary Fermi gas of  $^6\text{Li}$  atoms but extracted from the gas under harmonic confinement (filled squares with error bars). In the figure we include the results of our model, that is given by Eqs. (20) and (14) with both  $\lambda = 0.25$  (solid line) and  $\lambda = 0$  (dashed line). The figure shows

that in our model the gradient term, proportional to  $\lambda$ , plays a marginal role up to  $T/T_F \simeq 0.25$ . Above  $T/T_F \simeq 0.25$ , however, our results with  $\lambda = 0.25$  shows a better agreement with both Monte Carlo and experimental data than those with  $\lambda = 0$ . We stress that the gradient term is essential to describe accurately the zero-temperature surface effects of a trapped system, in particular with a small number of atoms, where the Thomas-Fermi (i.e.  $\lambda = 0$ ) approximation fails<sup>15</sup>. The value  $\lambda = 0.25$  gives the best fit of the Monte Carlo energy as a function of the particle number for  $\xi = 0.42$  (see Ref.<sup>15,9</sup> for details).

Our model does not show a phase transition. Nevertheless, the results of Fig. 1 strongly shows that our model works quite well not only in the superfluid regime, but also slightly above the critical temperature ( $T_c/T_F \simeq 0.15$ ) suggested by two theoretical groups.<sup>24,27</sup> This finding is not fully surprising. In presence of a pseudo-gap region, the temperature-dependent gap  $\Delta(T)$  of single-particle elementary excitations can be written as  $\Delta(T) = \Delta_{sc}(T) + \Delta_{pg}(T)$ , where  $\Delta_{sc}(T)$  is the superconducting gap and  $\Delta_{pg}(T)$  is the pseudogap.<sup>14</sup> At  $T_c$  the superconducting gap  $\Delta_{sc}(T)$  goes to zero, i.e.  $\Delta_{sc}(T_c) = 0$ , but the pseudo-gap  $\Delta_{pg}(T)$  remains finite and it becomes zero only at the higher temperature  $T^*$ .<sup>14</sup> For further details on the comparison between our model and other theories see Ref.<sup>9</sup>.

#### 4 Shear viscosity from thermodynamics

A first principle calculation of the shear viscosity is beyond the scope of the present work and we adopt the heuristic analysis of How and LeClair<sup>8</sup> to write it in terms of the scaled free energy  $\Phi(x)$  and its first derivative  $\Phi'(x)$ . The shear viscosity  $\eta$  can be estimated by using the formula<sup>28</sup>

$$\eta = \frac{1}{3} n m \bar{v} l_m, \quad (21)$$

where  $n$  is the total number density of the fluid,  $m$  is the mass of each particle in the fluid,  $\bar{v}$  is the average velocity of particles, and  $l_m$  is the length of the mean free path. The mean free path is written as

$$l_m = \frac{1}{n\sigma}, \quad (22)$$

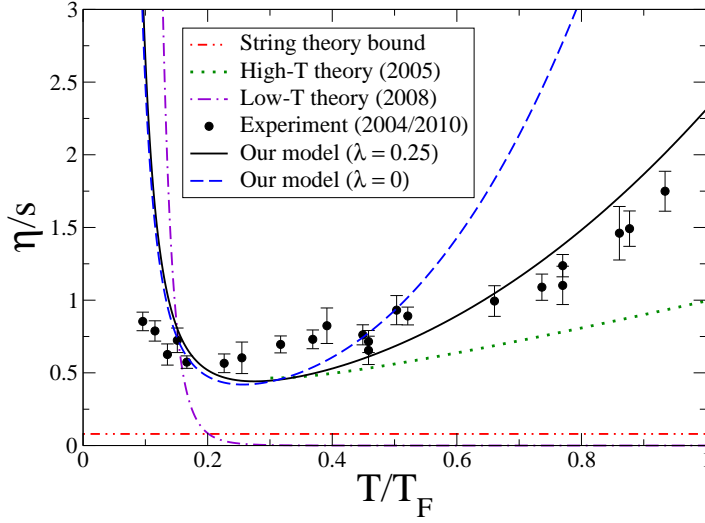
where  $\sigma$  is a suitable transport cross-section.<sup>28</sup>

For the unitary gas with two-spin-component fermions, the cross-section is given by

$$\sigma = \frac{4\pi}{|\mathbf{k}_1 - \mathbf{k}_2|^2}, \quad (23)$$

where  $\mathbf{k}_1 - \mathbf{k}_2$  is the relative wave number of two colliding fermions with opposite spin.<sup>12</sup> The average velocity  $\bar{v}$  of fermions can be related to the relative wave number  $|\mathbf{k}_1 - \mathbf{k}_2|$  by the formula<sup>8</sup>

$$\bar{v} = \sqrt{2} \frac{\hbar}{m} \langle |\mathbf{k}_1 - \mathbf{k}_2|^2 \rangle^{1/2}. \quad (24)$$



**Fig. 2** (Color online). Viscosity-entropy ratio  $\eta/s$  (in units of  $\hbar/k_B$ ) as a function of the scaled temperature  $T/T_F$ . Dot-dot-dashed line: the bound  $\eta/s = 1/(4\pi)$  predicted by field theory models inspired by string theory.<sup>3</sup> Filled circles with error bars: experimental data.<sup>4,7</sup> Dot-dashed line: low-temperature prediction of Ref.<sup>5</sup>. Dotted line: high-temperature prediction of Ref.<sup>6</sup>. Solid line: our model with  $\lambda = 0.25$ . Dashed line: our model with  $\lambda = 0$ . Other zero-temperature parameters of elementary excitations:  $\xi = 0.42$ ,  $\zeta = 0.9$ , and  $\gamma = 0.45$ .

In fact,  $\langle |\mathbf{k}_1 - \mathbf{k}_2|^2 \rangle = \langle k_1^2 + k_2^2 - 2\mathbf{k}_1 \cdot \mathbf{k}_2 \rangle = 2\bar{k}^2$ , because  $\langle \mathbf{k}_1 \cdot \mathbf{k}_2 \rangle = 0$  and  $\bar{k}^2 = \langle k_1^2 \rangle = \langle k_2^2 \rangle$ , and finally  $\bar{v} = (\hbar/m)\bar{k}$ .<sup>29</sup> In this way the shear viscosity becomes

$$\eta = \frac{m^3 \bar{v}^3}{6\pi\hbar^2}. \quad (25)$$

The average velocity  $\bar{v}$  can be estimated by imposing that the average kinetic energy is equal to the internal energy per particle,<sup>8</sup> namely

$$\frac{1}{2}m\bar{v}^2 = \frac{E}{N}. \quad (26)$$

By using Eq. (25) with  $\bar{v}$  given by Eq. (26) and  $E$  given by Eq. (20), the shear viscosity reads

$$\eta = n\hbar \frac{\pi}{2} \left( \Phi\left(\frac{T}{T_F}\right) - \frac{T}{T_F} \Phi'\left(\frac{T}{T_F}\right) \right)^{3/2}. \quad (27)$$

Notice that for  $T \rightarrow 0$ , the viscosity  $\eta$  goes to a constant value because also  $\sigma$  goes to a constant  $\simeq k_F^{-2}$ . This is in agreement with recent experimental results on the universal spin diffusion in a strongly interacting Fermi gas.<sup>30</sup> Finally, by considering Eq. (18) for the entropy of the unitary Fermi gas, we find that the viscosity-entropy ratio is given by

$$\frac{\eta}{s} = -\frac{\hbar}{k_B} \frac{\pi}{2} \frac{\left( \Phi\left(\frac{T}{T_F}\right) - \frac{T}{T_F} \Phi'\left(\frac{T}{T_F}\right) \right)^{3/2}}{\Phi'\left(\frac{T}{T_F}\right)}, \quad (28)$$

where  $s = S/V$  is the entropy density, i.e. the entropy per unit of volume. This formula gives the viscosity-entropy ratio in terms of the scaled free energy  $\Phi(x)$  and its first derivative  $\Phi'(x)$ . For  $T \rightarrow 0$  Eq. (28) gives  $\eta/s = +\infty$ . This divergence of  $\eta/s$  is a consequence of Eqs. (25) and (26) which impose, as previously stressed, a small but finite viscosity  $\eta$  while the entropy density  $s$  goes to zero.

In Fig. 2 we plot experimental data of the ratio  $\eta/s$  (filled circles with error bars). These data have been obtained by the group of Thomas<sup>4</sup> from the damping of radial breathing mode of the atomic cloud, and then elaborated by Schäfer and Chafin<sup>7</sup> with an energy-to-temperature calibration and averaging the local ratio  $\eta/s$  over the trap size. In the figure we insert also the bound from string theory (dot-dot-dashed line), the low-temperature prediction of Rupak and Schäfer<sup>5</sup> (dot-dashed line), and the high-temperature prediction of Bruun *et al.*<sup>6</sup> (dotted line). We plot also the results obtained with our model, Eq. (28) with Eq. (14), for  $\lambda = 0.25$  (solid line) and  $\lambda = 0$  (dashed line). The figure shows that our model is in good qualitative agreement with the experimental data up to  $T/T_F \simeq 0.4$ . Both with  $\lambda = 0.25$  (solid line) and  $\lambda = 0$  (dashed line) our model shows a minimum for  $\eta/s \simeq 0.44$  at  $T/T_F \simeq 0.27$ . Notice that the solid curve ( $\lambda = 0.25$ ) gives a reasonable agreement up to  $T/T_F \simeq 0.9$ .

We observe that the curve of  $\eta/s$  vs  $T/T_F$  obtained by How and LeClair,<sup>8</sup> on the basis of their version of Eq. (28) but with a very different procedure to calculate the scaled free energy  $\Phi(x)$ , does not seem to increase as  $T/T_F$  goes to zero. Actually, a very recent calculation<sup>31</sup> of the shear viscosity from current-current correlation functions suggests that  $\eta/s$  at low  $T$  becomes small rather than exhibiting the upturn. Nevertheless, the obtained theoretical values<sup>31</sup> of  $\eta/s$  appear quite large with respect to the experimental ones.

## 5 Conclusions

We have described the elementary excitations of the unitary Fermi gas as made of collective bosonic excitations and fermionic single-particle ones. We have obtained an analytical expression for the Helmholtz free energy, showing that it is reliable to study the low-temperature thermodynamics of the unitary Fermi system up to the critical temperature of the superfluid phase transition. By using this free energy and simple scaling arguments we have derived the viscosity-entropy ratio  $\eta/s$  as a function of the scaled temperature  $T/T_F$ . Contrary to other predictions, our curve of  $\eta/s$  vs  $T/T_F$  is in reasonable agreement with the available experimental data.

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